Structure of the Picard Stack and the Abel-Jacobi morphism for the curve

1

→ article "Simple connexité des fibres d'une application d'Abel-Jacobi et corps de classe local"

Prove the Conjecture for GL_1 using géo. Langlands methods.

Recall: X/k smooth projective curve. $\left[\pi_1(X)^{ab} \simeq \pi_1(\mathrm{Jac}(X))\right]$

$\mathcal{E}$ = rk. 1 $\overline{\mathbb{Q}_\ell}$ - local system on X

dual statement = géo. Langlands for GL_1

$d \geq 1$. $\mathrm{Div}^d = X^d/\sigma_d$ = Hilbert scheme of deg. d effective divisors on X

$\mathrm{Pic} = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}^d$ Picard scheme

$$\Sigma^d : (\mathrm{Div}^1)^d \twoheadrightarrow \mathrm{Div}^d$$

perverse in general but local sys. here since monodromy abelian

$$\mathcal{E}^{(d)} := \Sigma^d_* \left(\mathcal{E}^{\boxtimes d}\right)^{\sigma_d} = \text{rk. 1 } \overline{\mathbb{Q}_\ell} \text{ local system on } \mathrm{Div}^d$$

$AJ^d : Div^d \longrightarrow Pic^d$ Abel-Jacobi morphism

$D \longmapsto \mathcal{O}(D)$ Simply connected

Étale locally trivial fibration in projective spaces for $d>>0$

$\Rightarrow$ for $d>>0$ $\mathcal{E}^{(d)}$ descends to a rk. 1 $\overline{\mathbb{Q}_\ell}$-local system on Pic^d.

$\rightsquigarrow$ defines a rk. 1 $\overline{\mathbb{Q}_\ell}$-local system on $\coprod_{d>>0} Pic^d$

"Compatible with the monoid structure"

monoid generates $Pic = \coprod_{d \in \mathbb{Z}} Pic^d$

$\rightsquigarrow$ extends to Pic using the group structure.

Rem: $\underbrace{\mathcal{P}ic}_{\text{Picard stack}} \longrightarrow \underbrace{Pic}_{\text{Coarse moduli space}}$

$\mathbb{G}_m$-gerb (neutral if $X(k) \neq \phi$)

$\Rightarrow \pi_1(\mathcal{P}ic) = \pi_1(Pic)$

$\mathbb{G}_m$ connected

$\rightarrow$ Won't be the case in my framework: has to work with the Picard stack.

2

Recall:

$$E \begin{cases} \nearrow [E:\mathbb{Q}_p] < +\infty & \mathbb{F}_q = \mathcal{O}_E/\pi \\ \searrow E = \mathbb{F}_q((\pi)) \end{cases}$$

$\mathrm{Perf}_{\mathbb{F}_q} \ni S \rightsquigarrow X_S = Y_S/\varphi^{\mathbb{Z}}$ E-adic space

Ex: * $E = \mathbb{F}_q((\pi))$ $\quad Y_S = \mathbb{D}^*_S \circlearrowleft \varphi = \mathrm{Frob}_S$

$\{0 < |\pi| < 1\} \subset \mathbb{A}^1_S$

$$\begin{tikzcd} & \mathbb{D}^*_S \arrow[dl] \arrow[dr] & \\ S & & \mathbb{D}^*_{\mathbb{F}_q} = \mathrm{Spa}(E) \end{tikzcd}$$

* $S = \mathrm{Spa}(R, R^+)$ $\quad E|\mathbb{Q}_p$

$Y_S = \mathrm{Spa}\left(W_{\mathcal{O}_E}(R^+), W_{\mathcal{O}_E}(R^+)\right) \setminus V(\pi[\varpi])$

$\varpi \in R$ p.u.

$\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ + pro-étale topology

Rem: $\mathrm{Spa}(\overline{\mathbb{F}}_q)$ = final object of the pro-étale topos not representable by a perfectoid space

Pic = $\underbrace{\text{Picard stacks}}$ $\rightsquigarrow$ not completely evident

$$\mathrm{Pic}(S) = \{\text{line bundles}/X_S\}$$

Kedlaya-Liu: * $\mathcal{L}/X_S$ line bundle

$$|S| \longrightarrow \mathbb{Z}$$
$$s \longmapsto \deg\left(\mathcal{L}|X_{b(s),b(s)^+}\right)$$

is locally constant

$$\Rightarrow \mathrm{Pic} = \coprod_{d\in\mathbb{Z}} \underbrace{\mathrm{Pic}^d}_{\text{open/closed substacks}}$$

$$* \{\text{Line bundles}/X_S \ \overline{\text{fiberwise on } S} \text{ of deg. } 0\} \qquad \mathcal{L}$$

$$\downarrow \simeq \qquad\qquad \downarrow$$

$$\{\text{pro-étale rk. 1 } \underline{E}\text{-local systems}\} \qquad \text{"}H^0(\mathcal{L})\text{"}$$

relative coh. of $\mathcal{L}$

$\longrightarrow$ generalization of Lang ($\underline{\mathcal{O}_E}$-local systems)

$$T/S \longmapsto H^0(X_T, \mathcal{L}|_{X_T})$$

Thus: $\mathrm{Pic}^{0} \xrightarrow{\sim} \underbrace{\left[\mathrm{Spa}(\overline{\mathbb{F}}_q)/\underline{E}^{\times}\right]}_{\text{classifying stacks of pro-étale } \underline{E}^{\times}\text{-torsors.}}$

$$\mathcal{L} \longmapsto \underline{\mathrm{Isom}}(\mathcal{O}, \mathcal{L})$$

$\mathrm{Aut}(\mathcal{O}) = \underline{E}^{\times}$.

$* \ d \in \mathbb{Z}$ $\quad \underbrace{\mathcal{O}(d)}_{\text{depends on the choice of } \pi \text{ (fixed)}} = \text{deg. } d \text{ line bundle} / X_S$

$$Y_S \times_{\varphi^{\mathbb{Z}}} \mathbb{A}^1 \qquad \circlearrowleft \varphi \text{ acts like } \pi^{-d}.$$

$$\mathrm{Pic}^{d} \xrightarrow{\sim} \left[\mathrm{Spa}(\overline{\mathbb{F}}_q)/\underline{E}^{\times}\right]$$

$$\mathcal{L} \longmapsto \underline{\mathrm{Isom}}(\mathcal{O}(d), \mathcal{L})$$

Thus, coarse moduli space of $\mathrm{Pic}^{d} = \mathrm{Spa}(\overline{\mathbb{F}}_q)$ = trivial, no geometry. One has to work with the Picard stacks to have geometry.

Div^1:

$d \geq 1$. $\mathrm{Div}^d \longrightarrow \mathrm{Spa}(\overline{\mathbb{F}}_q)$ pro. étale sheaf.

Def: $\mathrm{Div}^d(S) = \left\{ (\mathcal{L}, u) \;\middle|\; \begin{array}{l} \mathcal{L} \text{ line bundle on } X_S \\ u \in H^0(X_S, \mathcal{L}) \text{ s.t. } \forall s \in S \\ u|_{X_{k(s), k(s)^+}} \neq 0 \end{array} \right\}$

$\breve{E} := \widehat{E^{un}}$

A construction:

sheaf on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$

$\mathrm{Spa}(\breve{E})$ when $E = \mathbb{F}_q((\pi))$ $= \mathrm{Spa}(\breve{E}^{1/p^\infty})$ as a sheaf on Perf

$$\mathrm{Spa}(E)^\diamond \longrightarrow \mathrm{Div}^1 \;\Big|\; \begin{array}{c} \| \\ \mathrm{Spa}(\overline{\mathbb{F}}_q((\pi^{1/p^\infty}))) \end{array}$$

sheaf of untilts to E $= \begin{cases} \text{diamond if } E|\mathbb{Q}_p \\ \text{perfectoid space if } E = \mathbb{F}_q((\pi)). \end{cases}$

$S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ $\quad S^\# / \mathrm{Spa}(E)$ an untilt.

"Cartier divisor" locally on S defined by $V(\xi)$

$$\begin{tikzcd} S^\# \arrow[r, hook] \arrow[dr] & X_S \arrow[d] \\ & \mathrm{Spa}(E) \end{tikzcd}$$

degree 1 primitive element.

$\rightsquigarrow$ defines the morphism $\mathrm{Spa}(E)^\diamond \longrightarrow \mathrm{Div}^1$

$\circlearrowright \varphi$ on $\mathrm{Spa}(E)^\diamond$; the morphism is φ-invariant since $S^\# \hookrightarrow Y_S \twoheadrightarrow X_S$

$$\boxed{\underline{\text{Prop}}: \quad \mathrm{Spa}(E)^\diamond/\varphi^{\mathbb{Z}} \xrightarrow{\sim} \mathrm{Div}^1}$$

What is this beast?

$\rightarrow$ this is a diamond: $\mathrm{Spa}(E)^\diamond \twoheadrightarrow \mathrm{Spa}(E)^\diamond/\varphi^{\mathbb{Z}}$ is a pro-étale presentation.

But this is <u>not spatial</u>: not quasiseparated since

$$\mathrm{Spa}(E)^\diamond \underset{\mathrm{Spa}(E)^\diamond/\varphi^{\mathbb{Z}}}{\times} \mathrm{Spa}(E)^\diamond = \underbrace{\coprod_{\mathbb{Z}} \mathrm{Spa}(E)^\diamond}_{\text{not q.c.}}$$

. That being said the morphism $\mathrm{Div}^1 \to \mathrm{Spa}(\overline{\mathbb{F}}_q)$ is relatively representable in locally spatial diamonds.

$\mathrm{Spa}(\overline{\mathbb{F}}_q)$: final object of the pro-étale topos, not representable i.e. $\forall S \in \mathrm{Perf}_{\mathbb{F}_q}$

$$\begin{array}{c} \mathrm{Div}^1_S = \text{locally spatial diamond} \\ \downarrow \\ S \end{array}$$

Ex: $E = \mathbb{F}_q((\pi))$

$$Y_S = \mathbb{D}^*_S \circlearrowleft \varphi_S \text{ and } \varphi_E$$

$$\begin{tikzcd} & Y_S \arrow[dl] \arrow[dr] & \\ S & & \mathbb{D}^*_{\mathbb{F}_q} = \mathrm{Spa}(E) \end{tikzcd}$$

$$\varphi_S\left(\sum_n x_n \pi^n\right) = \sum_n x_n^q \pi^n$$

$$\varphi_E\left(\sum_n x_n \pi^n\right) = \sum_n x_n \pi^{nq}$$

perfectoid open punctured disk

$$\begin{array}{c} X_S = \mathbb{D}^*_S / \varphi_S^{\mathbb{Z}} \\ \downarrow \\ \mathrm{Spa}(E) \end{array} \qquad \begin{array}{c} \mathrm{Div}^1_S = \mathbb{D}^{*,1/p^\infty}_{S} / \varphi_E^{\mathbb{Z}} \\ \downarrow \\ S \end{array}$$

* More generally $Y_S^\diamond = S \times \mathrm{Spa}(E)^\diamond$

$$X_S^\diamond = S \times \mathrm{Spa}(E)^\diamond / \varphi_S^{\mathbb{Z}} \quad \Big| \quad \mathrm{Div}^1_S = S \times \mathrm{Spa}(E)^\diamond / \varphi_{E^\diamond}^{\mathbb{Z}}$$

$\varphi_S \circ \varphi_{E^\circ}$ = absolute Frobenius of $Y_S^\diamond$
acts trivially on top. space or étale site

partial Frobenius.

$\Rightarrow \quad |X_S| = |\mathrm{Div}^1_S|$, X_S and Div^1_S have the same étale topos

Both are locally isomorphic but are not isomorphic.

$\Rightarrow$ different from the "usual case" "$\mathrm{Div}^1_X \neq X$".

Div^d: $\quad \Sigma^d : (\mathrm{Div}^1)^d \longrightarrow \mathrm{Div}^d$ morphism

of pro-étale sheaves on $\mathrm{Perf}_{\bar{\mathbb{F}}_q}$

not pro-étale.

Th: The morphism Σ^d is quasi-pro-étale surjective and this defines an isomorphism of diamonds

$$(\mathrm{Div}^1)^d / \mathfrak{S}_d \xrightarrow{\sim} \mathrm{Div}^d$$

pro-étale quotient

→ mainly a consequence of our factorization result with Fontaine for primitive elements in A_{inf}.

→ a morphism is quasi-pro-étale if pro-étale locally on the source and the target it is pro-étale.

Ex: $\mathbb{A}^{1,\diamond}_{\mathbb{Q}_q} \longrightarrow \mathbb{A}^{1,\diamond}_{\mathbb{Q}_q}$ is quasi-pro-étale

$z \mapsto z^n$ Kummer.

→ Scholze criterion: a qc morphism is quasi-pro-étale iff it has profinite geometric fibers.

The Abel-Jacobi morphism - $d \geq 1$

$AJ^d : \mathrm{Div}^d \longrightarrow \mathrm{Pic}^d = [\mathrm{Spa}(\overline{\mathbb{F}}_q) / \underline{E}^\times]$

$\mathbb{B}$ = sheaf on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ defined by $\mathbb{B}(S) = \mathcal{O}(Y_S)$

Fontaine's type period ring

$\mathbb{B}^{\varphi=\pi^d}$ = sheaf of global sections of $\mathcal{O}(d)$.

absolute Banach Colmez space

↳ not a BC. space, not even a diamond

but $\mathbb{B}^{\varphi=\pi^d}$ is relatively representable in BC spaces.

$$\begin{array}{c}\mathbb{B}^{\varphi=\pi^d}\\ \downarrow \\ \mathrm{Spa}(\overline{\mathbb{F}}_q)\end{array}$$

$\mathrm{Div}^d(S) = \{(\mathcal{L},u) \mid u \in H^0(X_S,\mathcal{L})$ non zero fiberwise/S ...$\}$

$$\Rightarrow \boxed{\mathrm{Div}^d = \left(\mathbb{B}^{\varphi=\pi^d}\setminus\{0\}\right)/\underline{E}^\times \xrightarrow{AJ^d} \left[\mathrm{Spa}(\overline{\mathbb{F}}_q)/\underline{E}^\times\right]}$$

[<u>Prop</u>: $\mathbb{B}^{\varphi=\pi^d}\setminus\{0\}$ is a locally spatial diamond]

Drinfeld module

<u>Ex</u>: $E=\mathbb{F}_q((\pi))$. $\mathcal{G}$ = π-divisible formal $\mathcal{O}_E$-module/$\overline{\mathbb{F}}_q$

given by

$$\mathcal{G} = \widehat{\mathbb{G}}^d_{a,\overline{\mathbb{F}}_q}, \quad \pi\cdot(x_0,\dots,x_{d-1}) = (F.x_{d-1}, x_0,\dots,x_{d-2})$$

Period isomorphism

$$\underbrace{\varprojlim_{\times\pi} \mathcal{G}}_{} \xrightarrow{\sim} \mathbb{B}^{\varphi=\pi^d}$$

formal E-vector space (as we called them wrt. Fontaine)

||

universal cover of $\mathcal{G}$

$$\mathrm{Spa}\left(\overline{\mathbb{F}}_q[[x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]]\right) \xrightarrow{\sim} \mathbb{B}^{\varphi=\pi^d}$$

$$(x_0, \dots, x_{d-1}) \longmapsto \sum_{i=0}^{d-1} \sum_{b\in\mathbb{Z}} x_i^{q^{-b}} \pi^{bd+i}$$

perfect adic space not perfectoid

$\to$ the point $x_0 = \dots = x_{d-1} = 0$ is not analytic.

$$\mathrm{Spa}\left(\overline{\mathbb{F}}_q[[x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]]\right) \setminus V(x_0, \dots, x_{d-1}) \xrightarrow{\sim} \mathbb{B}^{\varphi=\pi^d} \setminus \{0\}$$

||

$\mathrm{Spa}(-)_a$ = perfectoid space

$$\mathbb{B}^{\varphi=\pi^d} \setminus \{0\} = \bigcup_{i=0}^{d-1} D(\lambda_i) \simeq \mathbb{B}^{d-1,\,1/p^\infty}_{\overline{\mathbb{F}}_q((\lambda_i^{1/p^\infty}))}$$

$$\downarrow$$

$$\mathrm{Spa}\left(\overline{\mathbb{F}}_q((\lambda_i^{1/p^\infty}))\right)$$

= union of d perfectoid $(d-1)$-dimensional open balls over different perfectoid fields.

* [Thus: $\forall d \geq 1$, AJ^d is a pro-étale locally trivial fibration in $\mathbb{B}^{\varphi=\pi^d} \setminus \{0\}$.]

Geometric local class field theory

Starting point:

$$\breve{E} = \widehat{E^{un}} \circlearrowleft \sigma$$

$$\mathrm{Div}^1 = \mathrm{Spa}(E)^{\diamond}_{\overline{\mathbb{F}}_q} / \varphi^{\mathbb{Z}}$$

over $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$

$$\mathrm{Spa}(E)^{\diamond} \times_{\mathrm{Spa}(\overline{\mathbb{F}}_q)} \mathrm{Spa}(\overline{\mathbb{F}}_q) \circlearrowleft \varphi = \varphi_{E^{\diamond}} \times \mathrm{Id}.$$

$$\| $$

$$\mathrm{Spa}(\breve{E})^{\diamond}$$

$\overline{\mathbb{Q}}_\ell$-local systems on Div^1 = $\underset{\varphi_{E^{\diamond}}\times \mathrm{Id}}{\overset{\|}{\varphi}}$-equivariant $\overline{\mathbb{Q}}_\ell$-local systems on $\mathrm{Spa}(E)^{\diamond} \times \mathrm{Spa}(\overline{\mathbb{F}}_q)$

$\varphi_{E^{\diamond}} \times \sigma$ = absolute Frob acts trivially on étale site.

$\simeq$ $\mathrm{Id} \times \sigma$-equivariant $\overline{\mathbb{Q}}_\ell$-local systems on $\mathrm{Spa}(E)^{\diamond} \times \mathrm{Spa}(\overline{\mathbb{F}}_q)$

$=$ σ-equivariant $\overline{\mathbb{Q}}_\ell$-local systems on $\mathrm{Spa}(\breve{E})^{\diamond}$

$= \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_E)$

(8)

$\varphi: W_E \longrightarrow \overline{\mathbb{Q}}_\ell^\times$

$\rightsquigarrow \mathcal{E} =$ rk. 1 $\overline{\mathbb{Q}}_\ell$-local system on Div^1

$\rightsquigarrow \forall d \geq 1, \quad \mathcal{E}^{(d)} = \underbrace{\left(\Sigma_*^d\, \mathcal{E}^{\boxtimes d}\right)^{\sigma_d}}$ on Div^d

rk. 1 $\overline{\mathbb{Q}}_\ell$-local system using that $\mathrm{Div}^d = (\mathrm{Div}^1)^d / \sigma_d$

$\rightarrow$ this is where we use the preceding theorem.

[Th: $\forall d \geq 2$, $\mathbb{B}^{\varphi = \pi^d} \setminus \{0\}$ is simply connected: any finite étale cover is trivial.

needs to work "absolutely"

$\Uparrow$

$\triangle$: $\mathbb{B}^{\varphi=\pi^d}_{\mathbb{C}_p^\flat} \setminus \{0\}$ not simply connected]

$\Rightarrow \forall d \geq 2$, $\mathcal{E}^{(d)}$ descends to a local system on Pic^d.

$\mathrm{Pic}^d = [\cdot / \underline{E}^\times]$

i.e. is given by a character $E^\times \to \overline{\mathbb{Q}}_\ell^\times$

One checks this is equivalent to say that

$$\begin{tikzcd} W_E \arrow[rr, "\varphi"] \arrow[dr] & & \overline{\mathbb{Q}}_\ell^\times \\ & E^\times \arrow[ur] & \end{tikzcd}$$

given by Lubin-Tate

factorization

* $E = \overline{\mathbb{F}_q}((\pi))$

2- $\varinjlim_n \mathrm{Spa}\left(\overline{\mathbb{F}_q}[[x_0^{1/p^n}, \ldots, x_{d-1}^{1/p^n}]]\right) \setminus V(x_0, \ldots, x_{d-1})$

$\downarrow \simeq$ Elkik approximation (decompletion)

$\mathrm{Spa}\left(\overline{\mathbb{F}_q}[[x_0^{1/p^\infty}, \ldots, x_{d-1}^{1/p^\infty}]]\right) \setminus V(x_0, \ldots, x_{d-1})$

GAGA: A, I-adic - Noetherian

$\Rightarrow$

$$\mathrm{F.\acute{e}t}/\mathrm{Spec}(A) \setminus V(I) \xrightarrow{\sim} \mathrm{F.\acute{e}t.}/\underbrace{\mathrm{Spa}(A,A) \setminus V(I)}_{\mathrm{Spa}(A,A)_a}$$

$\rightarrow$ reduced to: $\mathrm{Spec}\left(\overline{\mathbb{F}_q}[[x_0, \ldots, x_{d-1}]]\right) \setminus V(x_0, \ldots, x_{d-1})$

is simply connected.

$\rightarrow$ Consequence of Zariski-Nagata purity

+ $\mathrm{Spec}\left(\overline{\mathbb{F}_q}[[x_0, \ldots, x_{d-1}]]\right)$ is simply connected (Hensel)

* $E|\mathbb{Q}_p$ - much more difficult - One reduces to prove ~~[illegible]~~

purity for $\mathbb{B}^{\varphi = \pi^d} \setminus \{0\} \hookrightarrow \mathbb{B}^{\varphi = \pi^d}$.