

Structure of the Picard Stack and the Abel-Jacobi morphism for the curve

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→ article "Simple connexité des fibres d'une application d'Abel-Jacobi et classe locale"

Prove the Conjecture for GL_1 using géo. Langlands methods.

Recall: X/k smooth projective curve. $[\pi_1(X)^{ab} \simeq \pi_1(\text{Jac}(X))]$

$\mathcal{E} = \text{nb. 1 } \overline{\mathbb{Q}_\ell}$ -local system on X

↓
dual statement = geo. Langlands for GL_1

$d \geq 1$. $\text{Div}^d = X^d / \sigma_d =$ Hilbert scheme of deg. d effective divisors on X

$\text{Pic} = \coprod_{d \in \mathbb{Z}} \text{Pic}^d$ Picard scheme

$\sum^d : (\text{Div}^1)^d \rightarrow \text{Div}^d$

$\mathcal{E}^{(d)} := \sum_*^d \left(\mathcal{E}^{\boxtimes d} \right)^{\sigma_d} = \text{nb. 1 } \overline{\mathbb{Q}_\ell}$ local system on Div^d

↙ reverse in general but local sys. here since monodromy abelian

$$AJ^d: \text{Div}^d \rightarrow \text{Pic}^d$$

Abel-Jacobi map

$$D \mapsto \mathcal{O}(D)$$

Simply connected

is a locally trivial fibration in projective spaces for $d \gg 0$

\Rightarrow for $d \gg 0$ $\mathcal{E}^{(d)}$ descends to a nb. 1 $\overline{\mathbb{Q}}_l$ -local system on Pic^d .

\rightsquigarrow defines a nb. 1 $\overline{\mathbb{Q}}_l$ -local system on Pic^d

"Compatible with the monoid structure"

monoid generates

$$\text{Pic} \cong \coprod_{d \geq 2} \text{Pic}^d$$

\rightsquigarrow extends to Pic using the group structure.

Picard stack

Coarse moduli space

$$\text{rem: } \overline{\text{Pic}} \rightarrow \text{Pic}$$

\uparrow G_m -gen (neutral if $X(b) \neq \emptyset$)

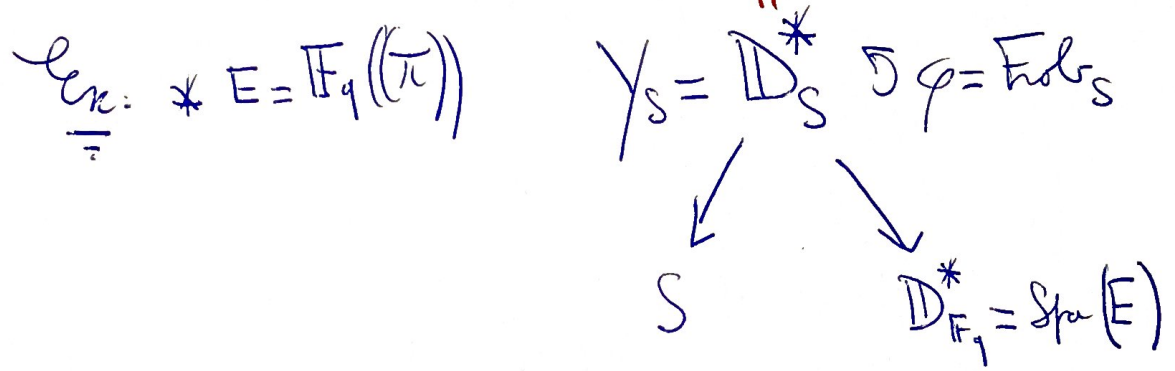
$$\Rightarrow \pi_1(\overline{\text{Pic}}) = \pi_1(\text{Pic})$$

\uparrow G_m connected

\rightarrow Won't be the case in my framework: has to work with the Picard stack.

Recall: $E \rightarrow [E: \mathbb{Q}_p] < +\infty \quad \mathbb{F}_q = \mathcal{O}_E / \pi$
 $E = \mathbb{F}_q((\pi))$

$\text{Perf}_{\mathbb{F}_q} \ni S \rightsquigarrow X_S = Y_S / \varphi^{\mathbb{Z}}$ E -adic space
 $\{0 < |a| < 1\} \subset \text{CA}_S^{\mathbb{Z}}$



$* S = \text{Spa}(R, R^+) \quad E | \mathbb{Q}_p$

$Y_S = \text{Spa}(W_{0,E}(R^+), W_{0,E}(R^+)) \setminus V(\pi[\omega])$
 $\omega \in R \text{ p.u.}$

$\text{Perf}_{\overline{\mathbb{F}_q}}$ + pro-étale topology

Rem: $\text{Spa}(\overline{\mathbb{F}_q}) =$ final object of the pro-étale topos
not representable by a perfectoid space

$\text{Pic} = \text{Picard stack}$

↳ not completely evident

$$\text{Pic}(S) = \{ \text{line bundles} / X_S \}$$

Kedlaya-Liu: * \mathcal{L}/X_S line bundle

$$|S| \rightarrow \mathbb{Z}$$

$$s \mapsto \deg(\mathcal{L}|_{X_{b(s), b(s)^+}})$$

is locally constant

$$\Rightarrow \text{Pic} = \coprod_{d \in \mathbb{Z}} \text{Pic}^d$$

open/closed substack

* $\{ \text{Line bundles} / X_S \text{ fiberwise on } S \text{ of deg. } 0 \}$ \mathcal{L}

$\downarrow \cong$
 $\{ \text{pro-stale nb. } \underline{\mathbb{E}}\text{-local systems} \}$ " $H^0(\mathcal{L})$ "

relative
coh. of \mathcal{L}

→ generalization of Lang
 $(\underline{C}_{\mathbb{E}}\text{-local systems})$

$$T/S \mapsto H^0(X_T, \mathcal{L}|_{X_T})$$

Thus: $\text{Pic}^0 \simeq [\text{Spa}(\overline{\mathbb{F}}_q) / \underline{E}^x]$

classifying stacks of pro-étale \underline{E}^x -torsors.

$$L \mapsto \underline{\text{Isom}}(O, L)$$

$$\text{Aut}(O) = \underline{E}^x$$

* $d \in \mathbb{Z}$

$O(d) =$ deg. d line bundle / X_S
 \cong \mathbb{P}^1 fact like π^{-d} .

depends on the choice of π (fixed)

$$Y_S \times_{\mathbb{A}^1} \mathbb{A}^1$$

$$\text{Pic}^d \simeq [\text{Spa}(\overline{\mathbb{F}}_q) / \underline{E}^x]$$

$$L \mapsto \underline{\text{Isom}}(O(d), L)$$

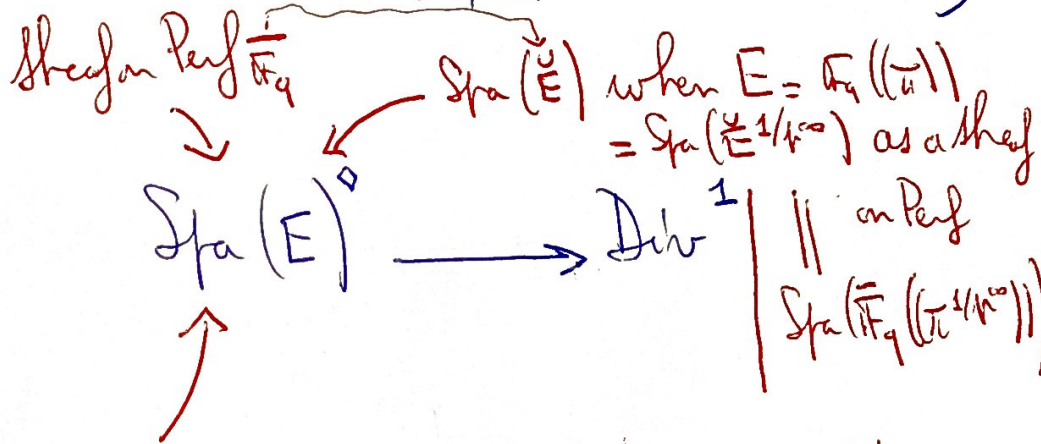
Thus, coarse moduli space of $\text{Pic}^d = \text{Spa}(\overline{\mathbb{F}}_q) =$ trivial, no geometry. One has to work with the Picard stacks to have geometry.

Div¹: $d \geq 1$. $\text{Div}^d \rightarrow \text{Spa}(\overline{\mathbb{F}_q})$ pro. étale sheaf.

Def: $\text{Div}^d(S) = \left\{ (L, u) \mid \begin{array}{l} L \text{ line bundle on } X_S \\ u \in H^0(X_S, L) \text{ s.t. } \forall \sigma \in S \\ u|_{X_{\mathbb{F}_q(\sigma), \mathbb{F}_q(\sigma)}} \neq 0 \end{array} \right\}$

$\check{E} := \widehat{E^{\text{un}}}$

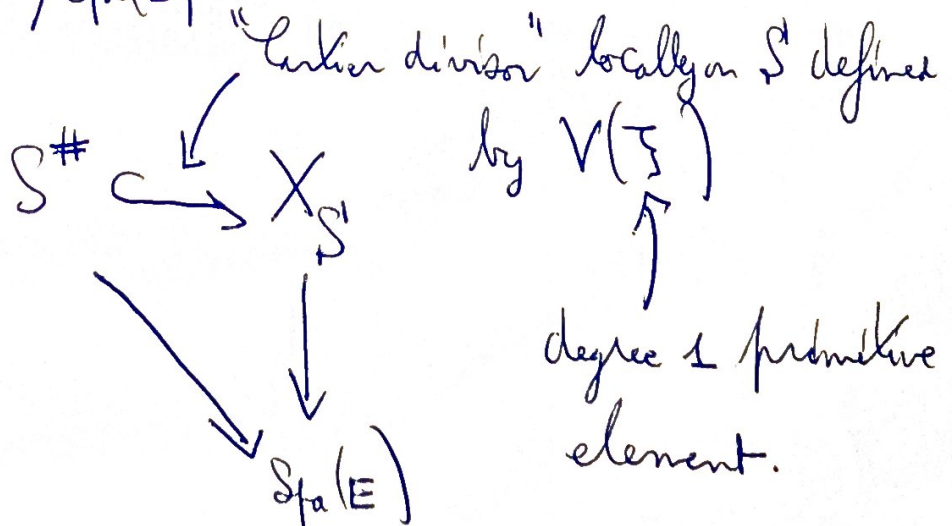
A Construction:



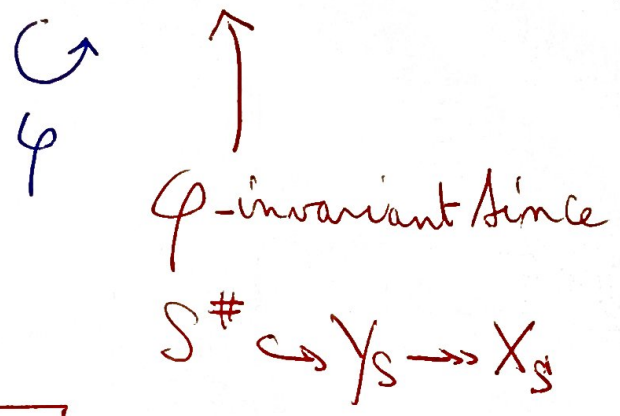
sheaf of unklts to $E \circlearrowleft = \left. \begin{array}{l} \text{diamond of } E/\mathbb{Q}_p \\ \text{perfectoid space of } E = \mathbb{F}_q((u)) \end{array} \right\}$

$S \in \text{Perf}_{\overline{\mathbb{F}_q}}$

$S^\# / \text{Spa}(E)$ an unkt.



defines the morphism $\text{Spa}(E)^\diamond \rightarrow \text{Div}^1$



Prop: $\text{Spa}(E)^\diamond / \phi^{\mathbb{Z}} \xrightarrow{\sim} \text{Div}^1$

What is this beast?

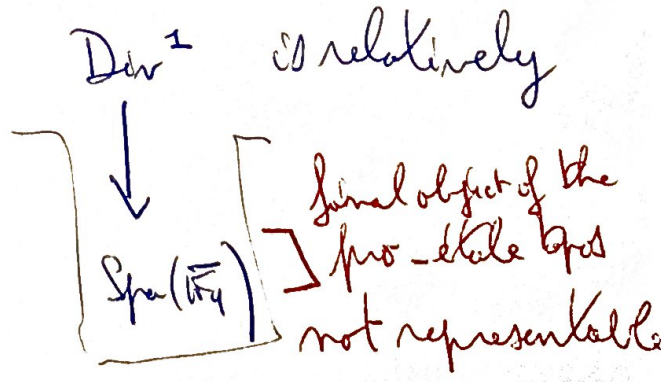
→ this is a diamond: $\text{Spa}(E)^\diamond \twoheadrightarrow \text{Spa}(E)^\diamond / \phi^{\mathbb{Z}}$ is a pro-étale presentation.

But this is not spatial: not quasi-separated since

$$\text{Spa}(E)^\diamond \times_{\text{Spa}(E)^\diamond / \phi^{\mathbb{Z}}} \text{Spa}(E)^\diamond = \coprod_{\mathbb{Z}} \text{Spa}(E)^\diamond$$

not q.c.

That being said the morphism Div^1 is relatively representable in locally spatial diamonds.



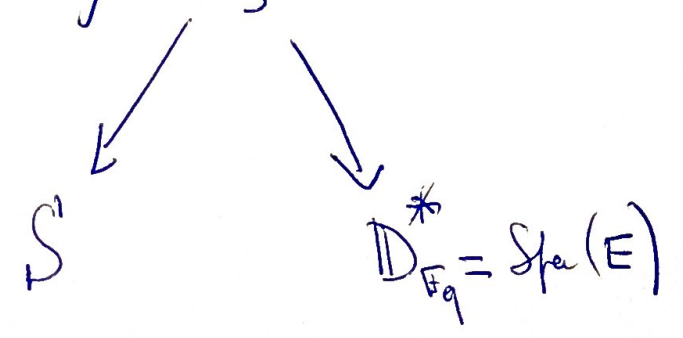
i.e. $V_S \vdash \text{Perf}_{\mathbb{F}_q}$

$\text{Div}_S^1 = \text{locally spatial diamond}$



Ex. $E = \mathbb{F}_q((\pi))$

$Y_S = \mathbb{D}_S^*$ $\cong \mathcal{G}_S \text{ and } \mathcal{G}_E$



$\mathcal{G}_S \left(\sum_n k_n \pi^n \right) = \sum_n k_n^q \pi^n$

$\mathcal{G}_E \left(\sum_n k_n \pi^n \right) = \sum_n k_n \pi^{nq}$

perfectoid open
punctured disks

$X_S = \mathbb{D}_S^* / \mathcal{G}_S^2$
↓
 $\text{Spa}(E)$

$\text{Div}_S^1 = \mathbb{D}_S^{*, \pm/\neq \infty} / \mathcal{G}_E^{\mathbb{Z}}$
↓
S

* More generally $Y_S^\diamond = S \times \text{Spa}(E)^\diamond$

$X_S^\diamond = S \times \text{Spa}(E)^\diamond / \mathcal{G}_S^2 \quad \Bigg| \quad \text{Div}_S^1 = S \times \text{Spa}(E)^\diamond / \mathcal{G}_{E^\diamond}^{\mathbb{Z}}$

$\varphi_S \circ \varphi_E =$ absolute Frobenius of Y_S^0
acts trivially on top space or étale site
↑ ↑
partial Frobenius.

$\Rightarrow |X_S| = |\text{Div}_S^1|$, X_S and Div_S^1 have the same étale topoi

Both are locally isomorphic but are not isomorphic.

→ different from the "usual case" " $\text{Div}_X^1 \cong X$ ".

Div^d : $\Sigma^d: (\text{Div}^1)^d \rightarrow \text{Div}^d$ morphism

of pro-étale sheaves on $\text{Perf}_{\overline{\mathbb{F}_q}}$ not pro-étale.

Th: The morphism Σ^d is quasi-pro-étale surjective and this defines an isomorphism of diamonds

$(\text{Div}^1)^d / \sigma_d \cong \text{Div}^d$
↑ pro-étale quotient

→ mainly a consequence of our factorization result with Fontaine for primitive elements in \mathbb{A}_q .

→ a morphism is quasi-pro-étale if pro-étale locally on the source and the target it is pro-étale.

$$\text{Ex: } A_{\mathbb{A}_q}^{1, \Delta} \rightarrow A_{\mathbb{A}_q}^{1, 0} \text{ is quasi-pro-étale}$$

$$z \mapsto z^n \text{ Kummer.}$$

→ Scholze criterion: a qc morphism is quasi-pro-étale iff it has profinite geometric fibers.

The Abel-Jacobi morphism - $d \geq 1$

$$AJ^d : \text{Div}^d \rightarrow \text{Pic}^d = [\text{St}_a(\overline{\mathbb{F}}_q) / \underline{\mathbb{E}}^x]$$

$$\mathbb{B} = \text{sheaf on } \text{Perf}_{\overline{\mathbb{F}}_q} \text{ defined by } \mathbb{B}(S) = \mathcal{O}(\gamma_S)$$

⌞
Fontaine's type period ring

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$B^{\varphi=\pi^d}$ = sheaf of global sections of $\mathcal{O}(d)$.

\perp

absolute Banach Colmez space

\hookrightarrow not a BC space, not even a diamond

but $B^{\varphi=\pi^d}$ is relatively representable in

\downarrow BC spaces.
 $\text{Spa}(\overline{\mathbb{F}_q})$

$\text{Div}^d(S) = \{ (Z, u) \mid u \in H^0(X_S, \mathcal{L}) \text{ non zero fiberwise / } S \}$

$$\Rightarrow \text{Div}^d = (B^{\varphi=\pi^d} \setminus \{0\}) / \mathbb{E}^\times \xrightarrow{AJ^d} [\text{Spa}(\overline{\mathbb{F}_q}) / \mathbb{E}^\times]$$

Prop: $B^{\varphi=\pi^d} \setminus \{0\}$ is a locally spatial diamond

\nearrow Dieudonné module

Ex: $E = \mathbb{F}_q((\pi))$.

$\mathcal{Y} = \pi$ -divisible formal \mathcal{O}_E -module

given by

$$\mathcal{Y} = \widehat{\mathbb{G}}_{a, \overline{\mathbb{F}_q}}^d, \quad \pi_*(\lambda_{0, \dots, \lambda_{d-1}}) = (F \cdot \lambda_{d-1}, \lambda_{0, \dots, \lambda_{d-2}})$$

Period isomorphism

$$\lim_{\leftarrow \times \pi} \mathcal{Y} \xrightarrow{\sim} \mathcal{B}^{\varphi = \pi^d}$$

formal \mathbb{E} -vector space (as we called them w.t. Fontaine)
 " universal cover of \mathcal{Y}

$$\mathrm{Spa}\left(\overline{\mathbb{F}_q} \llbracket \lambda_0^{1/t^\infty}, \dots, \lambda_{d-1}^{1/t^\infty} \rrbracket\right) \xrightarrow{\sim} \mathcal{B}^{\varphi = \pi^d}$$

$$(\lambda_0, \dots, \lambda_{d-1}) \mapsto \sum_{i=0}^{d-1} \sum_{b \in \mathbb{Z}} \lambda_i^{q^{-b}} \pi^{bd+i}$$

perfect adic space not perfectoid

→ the point $\lambda_0 = \dots = \lambda_{d-1} = 0$ is not analytic.

$$\mathrm{Spa}\left(\overline{\mathbb{F}_q} \llbracket \lambda_0^{1/t^\infty}, \dots, \lambda_{d-1}^{1/t^\infty} \rrbracket\right) \setminus V(\lambda_0, \dots, \lambda_{d-1}) \xrightarrow{\sim} \mathcal{B}^{\varphi = \pi^d} \setminus \{0\}$$

"
 $\mathrm{Spa}(-)_a =$ perfectoid space

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$$B^{\varphi=\pi^d} \setminus \{0\} = \bigcup_{i=0}^{d-1} D(x_i) \approx B_{\overline{\mathbb{F}_q}(x_i^{1/\pi^d})}^{d-1, 1/\pi^d}$$

$$\downarrow \\ \text{Spa}(\overline{\mathbb{F}_q}(x_i^{1/\pi^d}))$$

= union of d perfectoid $(d-1)$ -dimensional
open balls over different perfectoid fields.

* Thus: $\forall d \geq 1$, AJ^d is a pro-étale locally trivial
fibration in $B^{\varphi=\pi^d} \setminus \{0\}$.

Geometric local class field theory

Starting point:

~~XXXXXXXXXX~~

$$\hat{E} = \widehat{E^{an}} \circ \sigma$$

$$Div^1 = Spa(E)^\diamond / \varphi^\mathbb{Z} \quad \text{over } Perf_{\overline{\mathbb{F}_q}}$$

$$Spa(E)^\diamond \times_{Spa(\overline{\mathbb{F}_q})} Spa(\overline{\mathbb{F}_q}) \xrightarrow{\varphi} \varphi_{E^\diamond \times Id}$$

$$\cong Spa(\hat{E})^\diamond$$

$\overline{\mathbb{Q}_e}$ -local systems on $Div^1 = \varphi_{E^\diamond \times Id}$ -equivariant $\overline{\mathbb{Q}_e}$ -local systems on $Spa(E)^\diamond \times Spa(\overline{\mathbb{F}_q})$

$\varphi_{E^\diamond \times \sigma}$
"absolute Frobenius"
acts trivially on étale
site.

$\cong Id \times \sigma$ -equivariant $\overline{\mathbb{Q}_e}$ -local systems on $Spa(E)^\diamond \times Spa(\overline{\mathbb{F}_q})$

$= \sigma$ -equivariant $\overline{\mathbb{Q}_e}$ -local systems on $Spa(\hat{E})^\diamond$

$$= \text{Per}_{\overline{\mathbb{Q}_e}}(W_E)$$

~~W~~ $\varphi: W_E \rightarrow \overline{\mathbb{Q}_e}^x$

$\implies \mathcal{E} = \text{nb. 1 } \overline{\mathbb{Q}_e}\text{-local system on } \text{Div}^1$

$\implies \forall d \geq 1, \mathcal{E}^{(d)} = \left(\sum_{*}^d \mathcal{E}^{\otimes d} \right)^{\sigma_d}$ on Div^d

nb. 1 $\overline{\mathbb{Q}_e}$ -local system using that $\text{Div}^d = (\text{Div}^1)^d / \sigma_d$
 \rightarrow this is where we use the preceding theorem.

Th. $\forall d \geq 2, \mathbb{B}^{\varphi = \pi^d} \setminus \{0\}$ is simply connected: any finite étale cover is trivial. needs to work "absolutely"

Δ : $\mathbb{B}_{\mathbb{C}_h}^{\varphi = \pi^d} \setminus \{0\}$ not simply connected

$\implies \forall d \geq 2, \mathcal{E}^{(d)}$ descends to a local system on Pic^d .

\parallel
 $\left[\cdot / \overline{\mathbb{Q}_e}^x \right]$

i.e. is given by a character $E^x \rightarrow \overline{\mathbb{Q}_e}^x$

One checks this is equivalent to say that $W_E \xrightarrow{\varphi} \overline{\mathbb{Q}_e}^x$
given by $\swarrow \searrow$ E^x \nearrow factorization
Lubin-Tate

$$* E = \mathbb{F}_q((\pi))$$

$$2. \lim_{\substack{\longrightarrow \\ m}} \text{Spa}(\overline{\mathbb{F}_q}[[\pi^{1/m}, \dots, \pi^{1/m}]] \setminus V(\pi_0, \dots, \pi_{d-1}))$$

$$\downarrow \cong \quad \text{Ehlers approximation (decompletion)}$$

$$\text{Spa}(\overline{\mathbb{F}_q}[[\pi^{1/\infty}, \dots, \pi^{1/\infty}]] \setminus V(\pi_0, \dots, \pi_{d-1}))$$

GAGA: \mathbb{A}_1 -adic-Noetherian

$$\cong \text{F.ét}/\text{Spec}(A) \setminus V(I) \xrightarrow{\sim} \text{F.ét.} / \underbrace{\text{Spa}(A, A) \setminus V(I)}_{\text{Spa}(A, A)_a}$$

$$\rightarrow \text{reduced to: } \text{Spec}(\overline{\mathbb{F}_q}[[\pi_0, \dots, \pi_{d-1}]] \setminus V(\pi_0, \dots, \pi_{d-1}))$$

is simply connected.

\rightarrow consequence of Zariski-Nagata purity

+ $\text{Spec}(\overline{\mathbb{F}_q}[[\pi_0, \dots, \pi_{d-1}]] \setminus V(\pi_0, \dots, \pi_{d-1}))$ is simply connected (Hensel)

* E/\mathbb{Q}_p - much more difficult. One reduces to prove ~~the~~

purity for

$$B^{\varphi=\pi^d} \setminus \{0\} \hookrightarrow B^{\varphi=\pi^d}$$